Network Games with Many Attackers and Defenders (Formal Systems and Computality Theory)

数理解析研究所講究録
第 1729 巻 2011 年 146-151

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Abstract

In [2], Mavronicolas et al. presented a network game as an undirected graph whose nodes are exposed to infection by attackers, and whose edges are selectively protected by a defender. They showed that such a graph-theoretic game has no pure Nash equilibrium unless the graph is trivial. In [5] a network game is generalized to one with many defenders instead of a single player, so that there is a pure Nash equilibrium in more general situations. In this paper, we define a new model with the roles of players interchanged. As a result, we provide a graph theoretic characterization of Nash equilibrium of our new model.

1 Introduction

Game theory provides highly abstract tools which help us to represent and analyze interactive situations. Their abstraction allows us to apply them across disciplines, from set theory and logic to economic and biology. The most important notion in this approach is a Nash equilibrium which represents a situation where no player has anything to gain by changing his strategy unilaterally [3, 4].

Mavronicholas et al. [2] started a line of research of network games, that is, strategic games on graphs with two types of players, attackers and a defender. Attackers attack a *node* of the network to damage, and a defender is to protect the network by catching attackers in some part of the network. Each attacker wishes to maximize the number of successful (uncatched) attacks, whereas the defender aims at maximizing the number of attackers it catches. Mavronicolas et al. [2] first showed that no game with the defender playing a single edge (the Edge model) has a pure Nash equilibrium unless it is a trivial graph (see Theorem 4.3, [2]). They also defined a subclass of mixed Nash equilibria, called *matching* Nash equilibria, which are solvable in non-deterministic polynomial time for graphs with certain conditions (Theorem 5.3, [2]). Finally, they improved this non-deterministic algorithm into a deterministic polynomial-time algorithm for a bipartite graph (Theorem 6.3, [2]).

In [1], they considered a more general game, where the defender is able to scan a set of k links of the network. First, they showed that the existence problem of pure Nash equilibria is solvable in polynomial time. Then, they provided a graph-theoretic characterization of mixed Nash equilibria. Motivated by a class of polynomial time Nash equilibria, introduced for the Edge model, they defined a k-matching profile that generalizes a matching profile, and presented a polynomial time reduction for transforming any matching Nash equilibria of any instance of the Edge model to a k-matching Nash equilibria on a correspondence instance of their new model, and hence provided a characterization of graphs admitting k-matching Nash equilibria.

Inspired by these model, we believe that by increase the number of defenders will improve the quality of protection of the network. Therefore, in [5] we generalized their model by allowing many defenders instead of single player in order to increase the defenders' gain. In this paper we introduce a new type of network game where attackers aim to damage the network by attacking an edge, and defenders aim to protect the network by choosing a vertex. Both attackers and defenders make individual decisions for their placement in the network, seeking to maximize their objectives by pure strategies. We will discuss briefly the properties of graph admitting pure Nash equilibrium presented in [5] in Section 2. In Section 3, we introduced our new model of network game and provides a characterization of (pure) Nash equilibria.

2 Network Games with Many Defenders

We consider an undirected graph $G = (V, E)$ where V denotes a set of vertices and E denotes a set of edges. A *vertex cover* of G is a vertex set $C_V \subseteq V$ such that for each edge $(u, v) \in E$ either $u \in C_V$ or $v \in C_V$. A *minimum* vertex cover is one that has the minimum size. An *edge cover* of G is an edge set $C_E \subseteq E$ such that $\forall v \in V, \exists e = (v, u) \in C_E$. A *matching* M of G is a subset of E such that no vertex is incident to more than one edge in M (i.e. no two edges in M have a common vertex). A matching M is said to be *maximum* (size m) if for any other matching M', $m = |M| \geq |M'|$. A vertex set $I_V \subseteq V$ is an *independent set* of G if for all pairs of vertices $u, v \in I_V, (u, v) \notin E$. If v is incident of an edge e, then we write $v \in e$. We let $n_V(G)$ and $n_E(G)$ denote the number of vertices and edges in G. Whenever no confusion arises we write n_V and n_E instead of $n_V(G)$ and $n_E(G)$, respectively. Before we proceed to our new model, let see some characterization of (pure) Nash equilibrium in [5]. First, we see the *General Model* case defined as follows:

A game $\Gamma(G) = \langle \mathcal{N}, \mathcal{G} \rangle$ on graph G is defined by:

- $\mathcal{N} = \mathcal{N}_A \cup \mathcal{N}_D$ is the set of players where
  - $\mathcal{N}_A$ is a finite set of attackers (denoted by a_i, $1 \leq i \leq \nu$)
  - $\mathcal{N}_D$ is a finite set of defenders (denoted by d_j, $1 \leq j \leq \mu$)
- $\mathcal{G}$ is a collection of subgraphs of G, such that $G = \bigcup \mathcal{G}$
- The strategy set $\mathcal{S}$ of $\Gamma(G)$ is $V^\nu \times \mathcal{G}^\mu$

A profile $\boldsymbol{s}$ is an element of $\mathcal{S}$, i.e., $\boldsymbol{s} = \langle v_1, ..., v_\nu, S_1, ..., S_\mu \rangle \in \mathcal{S}$. Fix a profile $\boldsymbol{s}$. Define the payoff of the players as follows. The individual *Profit* ($P_{\boldsymbol{s}}$) of attacker a_i, $1 \leq i \leq \nu$, is given by

$$P_{\boldsymbol{s}}(a_i) = \begin{cases} 0 & \text{if } v_i \in S_j \text{ for some } j,\ 1 \le j \le \mu \\ 1 & \text{if } v_i \notin S_j \text{ for all } j,\ 1 \le j \le \mu \end{cases}$$

In other words, an attacker receives 0 if it is caught by a defender, and 1 otherwise. The individual $Profit$ ($P_{\boldsymbol{s}}$) of defender d_j, $1 \le j \le \mu$, is given by

$$P_{\boldsymbol{s}}(d_j) = |\{v : \exists i, 1 \le i \le \nu, v_i \in S_j \wedge v_i = v\}|$$

representing the number of attacked vertices it catches. Finally, we define $A_{\boldsymbol{s}} = \{v \in V : \exists i, 1 \le i \le \nu$, where $v = v_i\}$ and $D_{\boldsymbol{s}} = \{C \in \mathcal{G} : \exists j, 1 \le j \le \mu$ such that $C = S_j\}$.

Definition 2.1. *A profile $\boldsymbol{s}$ is a Nash equilibrium if for any player $r \in \{a_i, d_j\}$, $P_{\boldsymbol{s}}(r) \ge P_{\boldsymbol{s}'}(r)$ for any profiles $\boldsymbol{s}'$ which differs from $\boldsymbol{s}$ only on the strategy of r.*

In other words, in a Nash equilibrium no player can improve its individual profit by changing unilaterally his strategy.

Theorem 2.2. ([5]) *The game $\Gamma(G)$ has a Nash equilibrium if and only if there exist $D \subset \mathcal{G}$ and $A \subset V$ which satisfy the following conditions:*
(1) $|D| \le \mu$ and $|A| \le \nu$
(2) $\forall v \in V, \exists C \in D$ such that $v \in C$
(3) $\forall S \in D$, $|A \cap V(S)| = max_{C \in \mathcal{G}} |A \cap V(C)|$

For an *Edge Model* (i.e., $\mathcal{G} = E$), we have the following results.

Theorem 2.3. ([5]) *If the number of attackers ν is strictly less than n_V, then an edge model $\Gamma(G)$ has a Nash equilibrium $\boldsymbol{s}$ if and only if there exist D and A which satisfy the following conditions:*

- *Con0 $|D| \le \mu$ and $|A| \le \nu$*
- *Con1 D is an edge cover C_E of G*
- *Con2 A is a vertex cover C_V of the graph (V, D)*
- *Con3 A is an independent set I_V in G*

We have a particular case of edge model defined as follows.

Definition 2.4. *For a graph G, we have the following notations:*

- *$\Gamma_{max}(G)$ denotes the game $\Gamma(G)$ where $\nu = n_V - m$ and $\mu = n_E$*
- *$\Gamma_{min}(G)$ denotes the game $\Gamma(G)$ where $\nu = m$ and $\mu = n_V - m$*

where m is the size of a maximum matching in G.

We investigated the complexity of finding a Nash equilibrium in these particular games.

Theorem 2.5. ([5]) *Let G be a graph. Then, we can compute a Nash equilibrium of $\Gamma_{max}(G)$ in a non-deterministic polynomial time.*

Proof. Let A be a maximal independent set of vertices, and D the set of edges incident to a vertex in A. Then A and D satisfy the conditions of Theorem 2.3. □

In order to treat $\Gamma_{min}(G)$, we need a particular graph called a star graph. A star S_k is a complete bipartite graph $K_{1,k}$, that is a tree with one internal node and k leaves. First, we have the following definition.

Definition 2.6. *A graph G is said to have the property Prop(*) if and only if for a minimum edge cover C_E, there exists a map $f : V \to \{0,1\}$ such that*

- $\forall (u,v) \in E, f(u) + f(v) \geq 1$
- $\forall (u,v) \in C_E$, *we have* $f(u) + f(v) = 1$
- *for any multiple-edge star graph of C_E with a center u, $f(u) = 0$*

Theorem 2.7. ([5]) *A game $\Gamma_{min}(G)$ has a Nash equilibrium if and only if G satisfies the property Prop(*).*

If G is a bipartite graph or has no odd cycle, then the existence of such a map f is independent from the choice of the edge cover. Hence, a pure Nash equilibrium of $\Gamma_{min}(G)$ can be computed in polynomial time.

Theorem 2.8. ([5]) *If G is a bipartite graph or has no odd cycle, then deciding whether the game $\Gamma_{min}(G)$ has a Nash equilibrium is in polynomial time.*

3 New Strategic Model

As stated earlier, our new model is defined by interchanging the players' roles. We obtained a slightly different graph-theoretic characterization of Nash equilibria (Theorem 3.2).

Definition 3.1. *Let $G = (V,E)$ be an undirected graph with no isolated vertices. A new strategic game $\Gamma_{\alpha,\delta}(G) = \langle \mathcal{N}, \mathcal{S} \rangle$ on G is defined as follows:*

- $\mathcal{N} = \mathcal{N}_A \cup \mathcal{N}_D$ *is the set of players where*
  $\mathcal{N}_A$ *is a finite set of attackers* a_i, *where* $1 \leq i \leq \alpha$
  $\mathcal{N}_D$ *is a finite set of defenders* d_j, *where* $1 \leq j \leq \delta$
- $\mathcal{S} = E^\alpha \times V^\delta$ *is a strategy set of* $\Gamma_{\alpha,\delta}(G)$

A profile $\boldsymbol{s}$ is an element of $\mathcal{S}$, i.e., $\boldsymbol{s} = \langle e_1, ..., e_\alpha, v_1, ..., v_\delta \rangle \in \mathcal{S}$. Note that all players make their choice simultaneously. The number of attackers is α and the number of defenders is δ, for some fixed integers $\alpha, \delta \geq 1$. Now fix a profile $\boldsymbol{s}$ of the game $\Gamma_{\alpha,\delta}(G)$, we define the expected income (payoff) of the players as follows.

- The individual $Profit$ (P_s) of attacker a_i, $1 \le i \le \alpha$, is given by

$$P_s(a_i) = \begin{cases} 0 & \text{if } v_j \in e_i \text{ for some } j,\ 1 \le j \le \delta \\ 1 & \text{if } v_j \notin e_i \text{ for all } j,\ 1 \le j \le \delta \end{cases}$$

  In other words, an attacker receives 0 if it is caught by a defender, and 1 otherwise.

- The individual $Profit$ (P_s) of defender d_j, $1 \le j \le \delta$, is given by

$$P_s(d_j) = |\{e : \exists i, 1 \le i \le \alpha, v_j \in e_i \wedge e = e_i\}|$$

  representing the number of attacked edges it saves.

Before we proceed to the theorem, we define a set

$$A_s = \{e \in E : \exists i, 1 \le i \le \alpha,\ where\ e = e_i\}$$

$$D_s = \{v \in V : \exists j, 1 \le j \le \delta\ such\ that\ v = v_j\}$$

Theorem 3.2. *The game $\Gamma_{\alpha,\delta}(G)$ has a Nash equilibrium if and only if:*
(1) $|D| \le \delta$ and $|A| \le \alpha$
(2) $\forall e \in E, \exists v \in D$ such that $v \in e$
(3) $\forall v \in D$, $|A \cap E(v)| = max_{v \in V}|A \cap E(v)|$
where $E(v) = \{e \in E : v \in e\}$.

Proof. Suppose $\Gamma_{\alpha,\delta}(G)$ has a Nash equilibrium, say s. Let $A = A_s$ and $D = D_s$. Then, item (1) is straightforward. To prove (2), suppose that there exists $\bar{e} \in E$ such that $v \notin \bar{e}$ for all $v \in D$. Then, any attacker can receive 1 by switching to $\bar{e}$. Since s is a Nash equilibrium, each attacker must already gets 1, which means that all defenders receive 0. However, any defender can get at least 1 by switching to a vertex incident to an attacked edge, which contradicts to the assumption that s is Nash equilibrium. Similarly if (3) does not hold, there would be a $v_j \in D$ and v such that $|A \cap E(v_j)| < |A \cap E(v)|$. Thus, the defender j would find it beneficial to change his choice from v_j to v, which contradicts to the fact that s is Nash equilibrium.

Conversely, suppose there exist A and D satisfying condition (1), (2) and (3). By (1), let s be a profile so that each element of A (resp. D) is chosen by at least one attacker (resp. defender). By (2), no matter how an attacker changes his strategy, he will always get 0. Thus, an attacker has no motif to change his strategy. By (3), if a defender changes his choice, it won't increase his profit, since the number of protected edges is already maximum. □

Theorem 3.3. *If α is the size of a maximum matching in G and $\delta = 2\alpha$, then the game $\Gamma_{\alpha,\delta}(G)$ has a Nash equilibrium.*

Proof. Let A be a maximum matching in G and D be the set of vertices incident to an edge in A. Then obviously condition (1) and (3) of Theorem 3.2 hold. For (2), assume that there were $\bar{e} \in E$ such that $v \notin \bar{e}$ for all $v \in D$. Then, $A \cup \{\bar{e}\}$ is also a matching, which contradicts with the maximality of A. □

Definition 3.4. *The graph G is bipartite if $V = V_0 \cup V_1$ for some disjoint vertex sets $V_0, V_1 \subseteq V$ so that for each edge $(u,v) \in E$, $u \in V_0$ and $v \in V_1$ (or $u \in V_1$ and $v \in V_0$).*

Theorem 3.5. *For a bipartite graph G, a game $\Gamma_{\alpha,\delta}(G)$ has a Nash equilibrium if and only if $\alpha, \delta \geq m$, where m is the size of the maximum matching in G.*

Proof. The proof easily follows from König's duality theorem. For a bipartite graph G, if M is a maximum matching and C_V^{min} is a minimum vertex cover, then $\forall e \in M, \exists! v \in C_V$ such that $v \in e$. On the other hand, $\forall v \in C_V, \exists e \in M$ such that $v \in e$. So, $A = M, D = C_V^{min}$ satisfy the three conditions of Theorem 3.2. The other direction is similar. $\square$

Acknowledgment

The authors would like to thank Dr. MedSalem for encouragement and support.

References

[1] M. Gelastou, M. Mavronicolas, V. Papadopoulou, A. Philippou and P. Spirakis, The Power of the Defender, *CD-ROM Proceedings of the 2nd International Workshop on Incentive-Based Computing*, in conjunction with the *26th IEEE International Conference on Distributed Computing*, July 2006.

[2] M. Mavronicholas, V. Papadopoulou, A. Philippou, P. Spirakis. *A Network Game with Attackers and Defender*, Algorithmica **51**: 315-341, 2008.

[3] J.F. Nash. Equilibrium points in n-Person Game. In *Proceedings of the National Academy of Sciences of the United States of America*, **36**: 48-49, 1950.

[4] J.F. Nash. Non-cooperative Games. *Annals of Mathematics*, **54**(2): 286-295, 1951.

[5] MedYahya Ould-MedSalem, Y. Manoussakis, K. Tanaka. *A game on graphs with many non-centralized defenders*, (EURO XXII) 22nd European Conference on Operations Research, July 8-11, Prague, 2007.